The Mathematical Path

Factoring Polynomials: Difference of Squares

When we are factoring polynomials, there are certain patterns which can be immediately recognized and factored quickly. One such pattern is when you have exactly two terms, both of which are squares, and one is subtracted from the other. In this case, $a^2 - b^2 = (a+b)(a-b)$.

This module assumes you're familiar with:

- Introduction to Factoring Polynomials–Common Factor

If you are already familiar with expanding polynomials or factoring by grouping, you will find connections with that work, especially understanding why this pattern reliably factors the same way every time.

This module is a part of The Mathematical Path, which introduces mathematical ideas step-by-step to ensure that everyone's able to follow along. The latest version of the file can be obtained from `www.mathematicalpath.com/` along with other supporting materials.

Remember that mathematics builds upon itself, so if you're struggling with this material, it could be because you didn't quite understand something earlier on. Don't be afraid to back up and review some earlier mathematical concepts to get a stronger foundation and then come back to this module when you're ready. Visit The Mathematical Path website to trace back until you reach material you're comfortable with and then work your way back up to this module.

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In mathematics, there are certain patterns which can be recognized and handled specially, saving time and effort. These patterns do not obey special, different rules. Instead, they just fit a specific pattern that lets us skip certain steps in what would otherwise be a longer process.

One such pattern is when you have exactly two terms in a polynomial, both of which are squares, and one is subtracted from the other. When we see this pattern, we can immediately factor it as such:

$$a^2 - b^2 = (a+b)(a-b)$$

We will first practice identify polynomials which match the pattern to be factored according to this rule. Following this, we will demonstrate why this pattern works and apply it in practice.

Throughout this module we will generally only consider the positive square roots. We will show why we don't need to consider the negative square roots after prove the pattern always holds.

Identifying differences of squares

The pattern $a^2 - b^2$ is referred to as a "difference of squares," so named because each of the two terms are squares (that is, a result obtained from multiplying a value by itself), and one must be subtracted from the other.

The squares do not need to be a single number or variable, provided we can find the square root of the value. Recall that the square root of a product of terms is equal to product of the square roots of the terms: $\sqrt{xy} = \sqrt{x}\sqrt{y}$. For example, $9x^2$ is a square as its square roots are $3x$ and $-3x$. ()

We *can* find the square root of any positive value, but typically introducing radicals into a polynomial is not considered to be simplifying the expression. Therefore, we will focus only on cases where numeric coefficients have integer square roots and variables have an exponent which is a multiple of 2.

Problem set 1 (identifying differences of squares):
For each polynomial, identify if the polynomial matches the pattern to be a difference of squares (without requiring any additional factoring). If the polynomial does not match the pattern, explain why.

Sample problem: $16x^2 - 25y^4$

Sample solution: The term $16x^2$ is a square, as $\sqrt{16} = 4$ and $\sqrt{x^2} = x$ so $\sqrt{16x^2} = \sqrt{16}\sqrt{x^2} = 4x$. The term $25y^4$ is a square, as $\sqrt{25} = 5$ and $\sqrt{y^4} = y^2$ so $\sqrt{25y^4} = \sqrt{25}\sqrt{y^4} = 5y^2$. One of these terms is subtracted from the other. All the criteria are met to be a difference of squares.

a) $4 + 9x^2$

b) $100x^6 - 49y^2$

c) $4x^2 - 9y^2 - 16z^4$

d) $6x^2 - 9y^4$

e) $-9x^2 + 4y^6$

f) $36x^6 - 25y^5$

Why factoring differences of squares works

At the opening of this module, we made the claim that:

$$a^2 - b^2 = (a + b)(a - b)$$

Why is this true?

The easiest way to see this is true is probably to work backward from the given solution. We start with the factored form and expand it using expansion rules.

$$\begin{aligned} &(a + b)(a - b) \\ &= a^2 - ab + ba - b^2 \end{aligned}$$

Once we've expanded this, we see the expanded form contains the terms $-ab$ and $+ba$ (which is the same as $+ab$). No matter what the values of a and b are, these two terms cancel each other out because $-ab + ba = 0$. Substituting this result in, we get:

$$\begin{aligned} &a^2 - ab + ba - b^2 \\ &= a^2 + (-ab + ba) - b^2 \\ &= a^2 - b^2 \end{aligned}$$

This gives us our original difference of squares pattern. Since we have shown these two expressions are the same, we know this shortcut will work for any values of a and b.

This same sequence can be reversed if we want to show to this result directly. The bulk of this process is factoring by grouping, with one extra step to start things off. Just as we can multiply a value by 1 without changing its value, we can similarly add 0 to a value without changing its value. Since $ab - ab = 0$, we can add these two terms to the expression without changing its overall value.

$$\begin{aligned} &a^2 - ab \\ &= a^2 - b^2 + (ab - ab) \\ &= a^2 - b^2 + ab - ab \end{aligned}$$

At this point, we can proceed with our factoring by grouping.

$$\begin{aligned}
&a^2 - b^2 + ab - ab \\
&= (a^2 + ab) + (-b^2 - ab) \\
&= \frac{a}{a}(a^2 + ab) - \frac{-b}{-b}(-b^2 - ab) \\
&= a\left(\frac{a^2}{a} + \frac{ab}{a}\right) - b\left(\frac{-b^2}{-b} + \frac{-ab}{-b}\right) \\
&= a(a+b) - b(b+a) \\
&= \frac{a+b}{a+b}[a(a+b) - b(b+a)] \\
&= (a+b)\left[\frac{a(a+b)}{a+b} - \frac{b(b+a)}{a+b}\right] \\
&= (a+b)(a-b)
\end{aligned}$$

This directly gives us the factored form starting from the difference of squares.

Why we only need positive roots

Throughout this module we have only used the positive roots. But $x^2 = (-x)^2$, so why don't we need to worry about negative roots?

We already showed that $a^2 - b^2 = (a+b)(a-b)$ when we use the positive root. If we instead used the negative root of a^2, we can replace all instances of a in the factored version with $-a$: $a^2 - b^2 = [(-a) + b][(-a) - b]$. We now show that this is the same as $(a+b)(a-b)$. We begin by factoring -1 out from both sets of parentheses.

$$\begin{aligned}
&[(-a) + b][(-a) - b] \\
&= \frac{(-1)}{(-1)}[(-a) + b]\frac{(-1)}{(-1)}[(-a) - b] \\
&= (-1)\left[\frac{(-a)}{-1} + \frac{b}{-1}\right](-1)\left[\frac{(-a)}{-1} + \frac{(-b)}{-1}\right] \\
&= (-1)(a-b)(-1)(a+b)
\end{aligned}$$

We then re-arrange the terms and multiply the two -1 terms together to show that it is the same as if we assume the positive root of a^2.

$$\begin{aligned}
&(-1)(a-b)(-1)(a+b) \\
&= (-1)(-1)(a+b)(a-b) \\
&= (a+b)(a-b)
\end{aligned}$$

Therefore, whether we use the positive or negative root of a, we get the same factored form.

If we instead used the negative root of b^2, we can replace all instances of b in the factored version with $-b$. This immediately simplifies to the original factored form.

$$\begin{aligned} & [a+(-b)][a-(-b)] \\ = & (a-b)(a+b) \\ = & (a+b)(a-b) \end{aligned}$$

Since the result does not change if we use the negative root of either a^2 or b^2 individually, there is also no change if we use the negative roots of both a^2 and b^2.

Factoring by difference of squares

Having identified patterns where difference of squares factoring can be applied and proven to ourselves that the pattern works in all of these cases, we can apply it in practice.

Problem set 2 (applying difference of squares factoring):
For each polynomial, apply the difference of squares factoring pattern. If you need reassurance that the pattern works, either go through the same steps used to prove the general result above (adding and subtracting ab and using factoring by grouping), or expand your solution to check your result.

a) $9x^2 - 4$

b) $100x^6 - 49y^2$

c) $4x^2 - 9y^2$

d) $-9x^2 + 4y^6$

Combining grouping with common factors

When we combined factoring by grouping with common factoring, it did not matter which factoring we performed first. However, because of the restricted pattern we must match to perform factoring of a difference of squares, the order may or may not matter in this case. For example, $4x^2 - 16y^2$ is a difference of squares so the pattern can be applied directly, or we can first factor out the common factor 4. Conversely, $8x^2 - 18y^2$ is not a difference of squares, but a difference of squares patterns appears once the common factor of 2 is factored out.

Problem set 3 (combining the difference of squares with common factoring):
For each problem, use a combination of common factoring and factoring by the difference of squares to completely factor the polynomial.

a) $18x^2 - 32y^2$

b) $81x^2 - 9y^4$

c) $4x^3 - xy^2$

d) $4x^4 - 16x^2y^2$

Key Points

- A difference of squares is a specific pattern where:
  - there are two terms,
  - both terms are squares, and
  - one term is subtracted from the other.
- This can be factored as: $a^2 - b^2 = (a + b)(a - b)$.
- Recognizing a difference of squares is a shortcut that lets us skip intermediate steps; it doesn't obey different rules.
- A difference of squares pattern can appear in the middle of performing other factoring.

Next Steps

If you've read through this module, make sure you do all the exercises and check the solutions in the next section. If you got any answers wrong, go over the solutions and try to understand why you got a different answer.

If you are still feeling confused or don't understand something, please let us know. We're always willing to revisit and improve modules to ensure that everyone is able to follow along. It's not your fault for not understanding something, we just need to do a better job of presenting it! The best way for us to improve in order to help you and the next person who comes along is for you to tell us what didn't make sense. Similarly, if there's something you thought should be covered in this module but wasn't, be sure to let us know.

If you're feeling confident in this material, check out the www.mathematicalpath.com/ to find more modules that build on this one. If there's a module you want but doesn't exist yet, contact us to make a request and help us prioritize our efforts. We hope to see you soon as you continue down the Mathematical Path.

Solutions

Solutions to problem set 1

a) The term 4 is a square, as $\sqrt{4} = 2$. The term $9x^2$ is a square, as $\sqrt{9} = 3$ and $\sqrt{x^2} = x$ so $\sqrt{9x^2} = \sqrt{9}\sqrt{x^2} = 3x$. However, these terms are added, with neither subtracted from the other. Therefore, this polynomial is not a difference of squares (because the terms are added instead one being subtracted).

b) The term $100x^6$ is a square, as $\sqrt{100} = 10$ and $\sqrt{x^6} = x^3$ so $\sqrt{100x^6} = \sqrt{100}\sqrt{x^6} = 10x^3$. The term $49y^2$ is a square, as $\sqrt{49} = 7$ and $\sqrt{y^2} = y$ so $\sqrt{49y^2} = \sqrt{49}\sqrt{y^2} = 7y$. One of these terms is subtracted from the other. All the criteria are met to be a difference of squares.

c) The term $4x^2$ is a square, as $\sqrt{4} = 2$ and $\sqrt{x^2} = x$ so $\sqrt{4x^2} = \sqrt{4}\sqrt{x^2} = 2x$. The term $9y^2$ is a square, as $\sqrt{9} = 3$ and $\sqrt{y^2} = y$ so $\sqrt{9y^2} = \sqrt{9}\sqrt{y^2} = 3y$. The term $16z^2$ is a square, as $\sqrt{16} = 4$ and $\sqrt{z^2} = z$ so $\sqrt{16z^2} = \sqrt{16}\sqrt{z^2} = 4z$. However, a difference of squares requires exactly two terms, one of which is subtracted from the other. This polynomial has three terms. Therefore, this polynomial is not a difference of squares (because there are too many terms).

d) The term $6x^2$ is not a square, as 6 does not have an integer root. The term $9y^4$ is a square, as $\sqrt{9} = 3$ and $\sqrt{y^4} = y^2$ so $\sqrt{9y^4} = \sqrt{9}\sqrt{y^4} = 3y^2$. One of these terms is subtracted from the other. Therefore, this polynomial is not a difference of squares (because $6x^2$ is not a square).

e) The term $9x^2$ is a square, as $\sqrt{9} = 4$ and $\sqrt{x^2} = x$ so $\sqrt{9x^2} = \sqrt{9}\sqrt{x^2} = 3x$. The term $4y^2$ is a square, as $\sqrt{4} = 2$ and $\sqrt{y^2} = y$ so $\sqrt{4y^2} = \sqrt{4}\sqrt{y^2} = 2y$. While the operation between the two terms is addition we can re-arrange the polynomial so $-9x^2 + 4y^6 = 4y^6 - 9x^2$, which makes it clear that one of these terms is subtracted from the other. All the criteria are met to be a difference of squares.

f) The term $36x^6$ is a square, as $\sqrt{36} = 6$ and $\sqrt{x^6} = x^3$ so $\sqrt{36x^6} = \sqrt{36}\sqrt{x^6} = 6x^3$. The term $25y^5$ is not a square, as $\sqrt{25} = 5$ but $\sqrt{y^5}$ is not a square. One of these terms is subtracted from the other. Therefore, this polynomial is not a difference of squares (because $25y^5$ is not a square).

Solutions to problem set 2

a) The square root of $9x^2$ is $3x$. The square root of 4 is 2. Applying the difference of squares pattern gives a factored result of $(3x + 2)(3x - 2)$.

b) The square root of $100x^6$ is $10x^3$. The square root of $49y^2$ is $7y$. Applying the difference of squares pattern gives a factored result of $(10x^3 + 7y)(10x^3 - 7y)$.

c) The square root of $4x^2$ is $2x$. The square root of $9y^2$ is $3y$. Applying the difference of squares pattern gives a factored result of $(2x + 3y)(2x - 3y)$.

d) We begin by re-arranging the polynomial to match the difference of squares pattern: $-9x^2 + 4y^6 = 4y^6 - 9x^2$. The square root of $4y^6$ is $2y^3$. The square root of $9x^2$ is $3x$. Applying the difference of squares pattern gives a factored result of $(2y^3+3x)(2y^3-3x)$. Note that because of rearranging the terms to match the pattern, in this case it is the x term which is alternately added and subtracted in the two terms of the factor, not the y term.

Solutions to problem set 3

a) Neither $18x^2$ nor $32y^2$ are squares, but do share a common factor of 2.

$$18x^2 - 32y^2$$
$$= 2(9x^2 - 16y^2)$$

At this point, the term in parentheses, $(9x^2 - 16y^2)$, is a difference of squares, and we can factor it as such. The square root of $9x^2$ is $3x$. The square root of $16y^2$ is $4y$.

$$2(9x^2 - 16y^2)$$
$$= 2(3x + 4y)(3x - 4y)$$

Therefore, we can factor $18x^2 - 32y^2$ as $2(3x + 4y)(3x - 4y)$.

b) In this case, we have a common factor of 9, and both terms are squares, so we can start with either common factoring or factoring by a difference of squares. We show the results are the same in either case.

If we perform common factoring first, then we identify a common factor 9.

$$81x^2 - 9y^4$$
$$= 9(9x^2 - y^4)$$

The term in parentheses, $(9x^2 - y^4)$, is still a difference of squares, and we can factor it as such. The square root of $9x^2$ is $3x$. The square root of y^4 is y^2.

$$9(9x^2 - y^4)$$
$$= 9(3x + y^2)(3x - y^2)$$

Therefore, we can factor $81x^2 - 9y^2$ as $9(3x + y^2)(3x - y^2)$.

Alternatively, we can start with factoring the difference of squares. The square root of $81x^2$ is $9x$. The square root of $9y^4$ is $3y^2$.

$$81x^2 - 9y^4$$
$$= (9x + 3y^2)(9x - 3y^2)$$

Both of the terms $(9x + 3y^2)$ and $(9x - 3y^2)$ have a common factor of 3. We will factor these simultaneously.

$$(9x + 3y^2)(9x - 3y^2)$$
$$= 3(3x + y^2)3(3x - y^2)$$

However, we would typically not write 3×3 when factoring, but rather re-combine these factors to get 9.

$$3(3x + y^2)3(3x - y^2)$$
$$= 9(3x + y^2)(3x - y^2)$$

The result is the same as when common factoring was performed first.

c) Neither $4x^3$ nor xy^2 are squares, but do share a common factor of x.

$$4x^3 - xy^2$$
$$= x(4x^2 - y^2)$$

At this point, the term in parentheses, $(4x^2 - y^2)$, is a difference of squares, and we can factor it as such. The square root of $4x^2$ is $2x$. The square root of y^2 is y.

$$x(4x^2 - y^2)$$
$$= x(2x + y)(2x - y)$$

Therefore, we can factor $4x^3 - xy^2$ as $x(2x + y)(2x - y)$.

d) In this case, we have a common factor of $4x^2$ from each term, and both terms are squares, so we can start with either common factoring or factoring by a difference of squares. We show the results are the same in either case.

If we perform common factoring first, then we identify a common factor $4x^2$.

$$4x^4 - 16x^2y^2$$
$$= 4x^2(x^2 - 4y^2)$$

The term in parentheses, $(x^2 - 4y^2)$ is still a difference of squares, and we can factor it as such. The square root of x^2 is x. The square root of $4y^2$ is $2y$.

$$4x^2(x^2 - 4y^2)$$
$$= 4x^2(x + 2y)(x - 2y)$$

Therefore, we can factor $4x^4 - 16x^2y^2$ as $4x^2(x + 2y)(x - 2y)$.

Alternatively, we can start with factoring the difference of squares. The square root of $4x^4$ is $2x^2$. The square root of $16x^2y^2$ is $4xy$.

$$4x^4 - 16x^2y^2$$
$$= (2x^2 + 4xy)(2x^2 - 4xy)$$

Both of the terms $(2x^2 + 4xy)$ and $(2x^2 - 4xy)$ have a common factor of $2x$. We will factor these simultaneously.

$$(2x^2 + 4xy)(2x^2 - 4xy)$$
$$= 2x(x + 2y)2x(x - 2y)$$

Finally, we re-combine the two $2x$ factors to get $4x^2$.

$$2x(x + 2y)2x(x - 2y)$$
$$= 4x^2(x + 2y)(x - 2y)$$

The result is the same as when common factoring was performed first.