The Mathematical Path

Factoring Polynomials by Grouping

When we introduced factoring on polynomials, we relied on finding a factor which was shared by all the terms. If we don't have a single shared factor, there are other techniques we can use to factor a polynomial. This module introduces the technique of grouping, which can be applied to factor polynomials in certain situations.

This module assumes you're familiar with:

- Introduction to Factoring Polynomials–Common Factor

If you are already familiar with expanding polynomials, you will find connections with that work, but it's not essential to understand this material.

This module is a part of The Mathematical Path, which introduces mathematical ideas step-by-step to ensure that everyone's able to follow along. The latest version of the file can be obtained from `www.mathematicalpath.com/` along with other supporting materials.

Remember that mathematics builds upon itself, so if you're struggling with this material, it could be because you didn't quite understand something earlier on. Don't be afraid to back up and review some earlier mathematical concepts to get a stronger foundation and then come back to this module when you're ready. Visit The Mathematical Path website to trace back until you reach material you're comfortable with and then work your way back up to this module.

Be sure to visit `www.mathematicalpath.com/support/` to learn how you can support The Mathematical Path so we can create more modules!

When we first introduced factoring, we were searching for common factors (specifially the greatest common factor) which could be evenly divided every term of the polynomial. However, many polynomials will have a mix of terms which don't all share a single common factor. We still want to be able to factor these polynomials to gain insights into their properties, so we need to turn to other techniques. One of these alternate techniques is "grouping".

Grouping finds subsets of the terms of the polynomial which have a common factor, even if that factor is not shared by all the terms in the polynomial. We factor each of these subsets separately.

Grouping doesn't factor a polynomial fully as we still have an expression that is a sum. However, we can then attempt to find a common factor and complete the factor.

Grouping similar terms

We begin by examining the the polynomial and attempting to find ways that we can group the terms together so that each group has a common factor. Once we've grouped the terms, we perform common factoring on each of the groups separately.

As we group terms, make sure you don't leave a negative outside your grouping. If one or more of your terms is being subtracted, make sure you treat it as adding a negative term to group it properly. We'll see an example of this in the first sample problem.

Problem set 1 (grouping similar terms):

For each polynomial, divide the terms into groups where each group shares a common factor. Then do common factoring on each group.

Sample problem: $10xy + 15x - 4y - 6$

Sample solution: There is more than one way to group terms in ways that share factors.

One way is to group $10xy$ with $15x$, leaving the other group being $-4y$ and -6. We'll show these groups with parentheses.

$$(10xy + 15x) + (-4y - 6)$$

In this case, $(10xy + 15x)$ has a common factor of $5x$. The common factor of $(-4y-6)$ is -2. We do common factoring on each of these groups separately:

$$\begin{aligned}&\frac{5x}{5x}(10xy + 15x)\\ =&\ 5x\left(\frac{10xy}{5x} + \frac{15x}{5x}\right)\\ =&\ 5x(2y + 3)\end{aligned}$$

and

$$\begin{aligned}&\frac{-2}{-2}(-4y - 6)\\ =&\ -2\left(\frac{-4y}{-2} + \frac{-6}{-2}\right)\\ =&\ -2(2y + 3)\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}&10xy + 15x - 4y - 6\\ &= (10xy + 15x) + (-4y - 6)\\ &= 5x(2y + 3) - 2(2y + 3)\end{aligned}$$

Another way is to group $10xy$ with $-4y$, leaving the other group being $15x$ and -6. We'll show these groups with parentheses.

$$(10xy - 4y) + (15x - 6)$$

In this case, $(10xy - 4y)$ has a common factor of $2y$. The common factor of $(15x - 6)$ is 3. We do common factoring on each of these groups separately:

$$\begin{aligned}&\frac{2y}{2y}(10xy - 4y)\\ &= 2y\left(\frac{10xy}{2y} + \frac{-4y}{2y}\right)\\ &= 2y(5x - 2)\end{aligned}$$

and

$$\begin{aligned}&\frac{3}{3}(15x - 6)\\ &= 3\left(\frac{15x}{3} + \frac{-6}{3}\right)\\ &= 3(5x - 2)\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}&10xy + 15x - 4y - 6\\ &= (10xy - 4y) + (15x - 6)\\ &= 2y(5x - 2) + 3(5x - 2)\end{aligned}$$

There is one final way to group the terms, grouping $10xy$ with -6 and $-4y$ with $15x$. However, while $10xy$ and -6 have a common factor of 2, there is no common factor between $-4y$ and $15x$. We will see in the next problem set why this choice of grouping is not as useful as the other two. For now, it is sufficient to note that we were unable to perform as much factoring on this grouping as we could in either of the other two.

a) $21xy + 14x + 3y + 2$

b) $12x^2z + 30x^2 - 14yz - 35y$

c) $27ax+18ay+36az+15x+10y+20z$

d) $5xy - 5ax - 3yz + 3az$

Factoring by grouping

In these exercises, after we have grouped the terms, found the greatest common factor of each group, and factored each group, we notice something interesting. After we've factored out the common factor from each group, the part that is left is the same in every group for for each polynomial! Now we can perform common factoring.

When we perform common factoring on the groups, we need to choose if we are factoring out the positive or negative version of the greatest common factor. Once you've done factoring on one group, choose the positive or negative option for the remaining groups that will ensure what's left after factoring matches your first group.

It is important to note that not every polynomial can be factored this way. In many cases, grouping terms and factoring them will not give a result where all groups share a term. However, when it is possible, factoring by grouping can produce fully factored polynomials.

Problem set 2 (factoring the common polynomial):
For each exercise from the previous problem set, factor the common polynomial factor from each group to fully factor the expression.

Sample problem: $10xy + 15x - 4y - 6$

Sample solution: Two groupings were previous considered. We will finish factoring both of them, and show they give the same result.

$5x(2y + 3) - 2(2y + 3)$: Both terms have a factor $(2y + 3)$. We can factor this value out of each term using the common factoring process.

$$\begin{aligned}
&\frac{(2y+3)}{(2y+3)}\left[5x(2y+3) - 2(2y+3)\right] \\
&= (2y+3)\left[\frac{5x(2y+3)}{(2y+3)} + \frac{(-2)(2y+3)}{(2y+3)}\right] \\
&= (2y+3)(5x-2)
\end{aligned}$$

$2y(5x-2)+3(5x-2)$ Both terms have a factor $(5x-2)$. We can factor this value out of each term using the common factoring process.

$$\begin{aligned}&\frac{(5x-2)}{(5x-2)}\left[2y(5x-2)+3(5x-2)\right]\\&=(5x-2)\left[\frac{2y(5x-2)}{(5x-2)}+\frac{3(5x-2)}{(5x-2)}\right]\\&=(5x-2)(2y+3)\end{aligned}$$

In the previous problem set, we did not show any preference between these two groupings. We can see now why this is the case, as both groupings resulted in the same final factored expression: $(2y+3)(5x-2)=(5x-2)(2y+3)$.

This also demonstrates why the final grouping was not as optimal.

$$\begin{aligned}&(10xy-6)+(-4y+15x)\\&=2(5xy-3)+(15x-4y)\end{aligned}$$

There is no common factor here which can be factored out. Therefore, either of the other two groupings are preferable in order to get a fully factored polynomial.

a) $21xy+14x+3y+2$

b) $12x^2z+30x^2-14yz-35y$

c) $27ax+18ay+36az+15x+10y+20z$

d) $5xy-5ax-3yz+3az$

Combining grouping with common factors

We can apply both factoring by grouping and common factoring to the same expression. Either factoring operation can be applied first, and we will obtain the same result in either case. This is because multiplication is commutative. That is, the order that we do multiplication does not make a difference to the overall result.

Problem set 3 (factoring the common polynomial):
For each expression, apply factoring both common factoring and factoring by grouping.

Sample problem: $36x+20xz-18y^2-10y^2z$

Sample solution: We may notice first that each term of this expression is even. Therefore, there is a common factor of 2 which can be factored out.

$$\begin{aligned}
&36x + 20xz - 18y^2 - 10y^2z \\
&= \frac{2}{2}(36x + 20xz - 18y^2 - 10y^2z) \\
&= 2\left(\frac{36x}{2} + \frac{20xz}{2} + \frac{-18y^2}{2} + \frac{-10y^2z}{2}\right) \\
&= 2(18x + 10xz - 9y^2 - 5y^2z)
\end{aligned}$$

We then apply factoring by grouping to the portion of the expression in parentheses.

$$\begin{aligned}
&18x + 10xz - 9y^2 - 5y^2z \\
&= (18x + 10xz) + (-9y^2 - 5y^2z)
\end{aligned}$$

We identify common factors of $2x$ and $-y^2$ for the two groups respectively:

$$\begin{aligned}
&(18x + 10xz) + (-9y^2 - 5y^2z) \\
&= \frac{2x}{2x}(18x + 10xz) + \frac{-y^2}{-y^2}(-9y^2 - 5y^2z) \\
&= 2x\left(\frac{18x}{2x} + \frac{10xz}{2x}\right) + (-y^2)\left(\frac{-9y^2}{-y^2} + \frac{-5y^2z}{-y^2}\right) \\
&= 2x(9 + 5z) + (-y^2)(9 + 5z)
\end{aligned}$$

We then factor the common term $(9 + 5z)$:

$$\begin{aligned}
&2x(9 + 5z) + (-y^2)(9 + 5z) \\
&= (9 + 5z)(2x - y^2)
\end{aligned}$$

Alternatively, we can group the terms a different way.

$$\begin{aligned}
&18x + 10xz - 9y^2 - 5y^2z \\
&= (18x - 9y^2) + (10xz - 5y^2z) \\
&= \frac{9}{9}(18x - 9y^2) + \frac{5z}{5z}(10xz - 5y^2z) \\
&= 9\left(\frac{18x}{9} + \frac{-9y^2}{9}\right) + 5z\left(\frac{10xz}{5z} + \frac{-5y^2z}{5z}\right) \\
&= 9(2x - y^2) + 5z(2x - y^2) \\
&= (2x - y^2)(9 + 5z)
\end{aligned}$$

We combine these results to obtain a fully factored term expression:

$$\begin{aligned} & 2(18x + 10xz - 9y^2 - 5y^2z) \\ = & \, 2(2x - y^2)(9 + 5z) \end{aligned}$$

Instead of beginning with the common factor, we could have started with factoring by grouping. There are two ways we can group the terms.

$$\begin{aligned} & 36x + 20xz - 18y^2 - 10y^2z \\ = & \, (36x + 20xz) + (-18y^2 - 10y^2z) \\ = & \, \frac{4x}{4x}(36x + 20xz) + \frac{-2y^2}{-2y^2}(-18y^2 - 10y^2z) \\ = & \, 4x\left(\frac{36x}{4x} + \frac{20xz}{4x}\right) + (-2y^2)\left(\frac{-18y^2}{-2y^2} + \frac{-10y^2z}{-2y^2}\right) \\ = & \, 4x(9 + 5z) + (-2y^2)(9 + 5z) \\ = & \, (9 + 5z)(4x - 2y^2) \end{aligned}$$

or

$$\begin{aligned} & 36x + 20xz - 18y^2 - 10y^2z \\ = & \, (36x - 18y^2) + (20xz - 10y^2z) \\ = & \, \frac{18}{18}(36x - 18y^2) + \frac{10z}{10z}(20xz - 10y^2z) \\ = & \, 18\left(\frac{36x}{18} + \frac{-18y^2}{18}\right) + 10z\left(\frac{20xz}{10z} + \frac{-10y^2z}{10z}\right) \\ = & \, 18(2x - y^2) + 10z(2x - y^2) \\ = & \, (2x - y^2)(18 + 10z) \end{aligned}$$

Once we have performed the factoring by grouping, we perform common factoring on one of the two terms. If we chose the first grouping:

$$\begin{aligned} & (4x - 2y^2) \\ = & \, \frac{2}{2}(4x - 2y^2) \\ = & \, 2\left(\frac{4x}{2} + \frac{-2y^2}{2}\right) \\ = & \, 2(2x - y^2) \end{aligned}$$

Or if we chose the second grouping:

$$\begin{aligned}
&(18 + 10z)\\
&= \frac{2}{2}(18 + 10z)\\
&= 2\left(\frac{18}{2} + \frac{10z}{2}\right)\\
&= 2(9 + 5z)
\end{aligned}$$

Combining these two results and re-ordering the terms, we obtain $2(2x - y^2)(9+5z)$ in either case, the same result as when we performed the common factoring first.

a) $12xy - 15xz + 36y - 45z$

b) $20xz + 45x + 12xyz + 27xy$

Key Points

- Factoring by grouping consists of the following steps:
  - Arrange terms into groups that each have a common factor.
  - Perform common factoring on each group.
  - If what remains of the group after factoring the common factor is the same for each group, perform common factoring on this value across all the groups.
- The same factored result can be obtained by different ways of grouping the terms.
- Not all expressions can be factored this way.
- If all terms share a common factor, the common factoring and factoring by grouping can be performed in either order.

Next Steps

If you've read through this module, make sure you do all the exercises and check the solutions in the next section. If you got any answers wrong, go over the solutions and try to understand why you got a different answer.

If you are still feeling confused or don't understand something, please let us know. We're always willing to revisit and improve modules to ensure that everyone is able to follow along. It's not your fault for not understanding something, it's our fault it wasn't presented clearly enough! The best way for us to improve in order to help you and the next person who comes along is for you to tell us what didn't make sense. Similarly, if there's something you thought should be covered in this module but wasn't, be sure to let us know.

If you're feeling confident in this material, check out the `www.mathematicalpath.com/` to find more modules that build on this one. If there's a module you want but doesn't exist yet, contact us to make a request and help us prioritize our efforts. We hope to see you soon as you continue down the Mathematical Path.

Solutions

Solutions to problem set 1

a) We first consider this grouping:

$$(21xy + 14x) + (3y + 2)$$

The group $(21xy + 14x)$ has a common factor of $7x$. The group $(3y + 2)$ has a common factor of 1. We do common factoring on each of these groups separately:

$$\begin{aligned} &\frac{7x}{7x}(21xy + 14x) \\ &= 7x\left(\frac{21xy}{7x} + \frac{14x}{7x}\right) \\ &= 7x(3y + 2) \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{1}(3y + 2) \\ &= 1\left(\frac{3y}{1} + \frac{2}{1}\right) \\ &= 1(3y + 2) \end{aligned}$$

While we could simplify this by eliminiating the 1 factor, we'll keep the same format as the other group. Putting this together, we have:

$$\begin{aligned} &21xy + 14x + 3y + 2 \\ &= (21xy + 14x) + (3y + 2) \\ &= 7x(3y + 2) + 1(3y + 2) \end{aligned}$$

Alternatively, we can apply this grouping:

$$(21xy + 3y) + (14x + 2)$$

The group $(21xy + 3y)$ has a common factor of $3y$. The group $(14x + 2)$ has a common factor of 2. We do common factoring on each of these groups separately:

$$\begin{aligned} &\frac{3y}{3y}(21xy + 3y) \\ &= 3y\left(\frac{21xy}{3y} + \frac{3y}{3y}\right) \\ &= 3y(7x + 1) \end{aligned}$$

and

$$\begin{aligned} &\frac{2}{2}(14x + 2) \\ &= 2\left(\frac{14x}{2} + \frac{2}{2}\right) \\ &= 2(7x + 1) \end{aligned}$$

Putting this together, we have:

$$\begin{aligned} &21xy + 14x + 3y + 2 \\ &= (21xy + 3y) + (14x + 2) \\ &= 3y(7x + 1) + 2(7x + 1) \end{aligned}$$

This final grouping:

$$(21xy + 2) + (14x + 3y)$$

has no common factors.

b) We first consider this grouping:

$$(12x^2z + 30x^2) + (-14yz - 35y)$$

The group $(12x^2z + 30x^2)$ has a common factor of $6x^2$. The group $(-14yz - 35y)$ has a common factor of $-7y$. We do common factoring on each of these groups separately:

$$\begin{aligned} &\frac{6x^2}{6x^2}(12x^2z + 30x^2) \\ &= 6x^2\left(\frac{12x^2z}{6x^2} + \frac{30x^2}{6x^2}\right) \\ &= 6x^2(2z + 5) \end{aligned}$$

and

$$\begin{aligned}
&\frac{-7y}{-7y}(-14yz-35y)\\
=& -7y\left(\frac{-14yz}{-7y}+\frac{-35y}{-7y}\right)\\
=& -7y(2z+5)
\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}
&12x^2z+30x^2-14yz-35y\\
&=(12x^2z+30x^2)+(-14yz-35y)\\
&=6x^2(2z+5)-7y(2z+5)
\end{aligned}$$

Alternatively, we can apply this grouping:

$$(12x^2z-14yz)+(30x^2-35y)$$

The group $(12x^2z-14yz)$ has a common factor of $2z$. The group $(30x^2-35y)$ has a common factor of 5. We do common factoring on each of these groups separately:

$$\begin{aligned}
&\frac{2z}{2z}(12x^2z-14yz)\\
&=2z\left(\frac{12x^2z}{2z}+\frac{-14yz}{2z}\right)\\
&=2z(6x^2-7y)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{5}{5}(30x^2-35y)\\
&=2\left(\frac{30x^2}{5}+\frac{-35y}{5}\right)\\
&=5(6x^2-7y)
\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}
&12x^2z+30x^2-14yz-35y\\
&=(12x^2z-14yz)+(30x^2-35y)\\
&=2z(6x^2-7y)+5(6x^2-7y)
\end{aligned}$$

The final grouping:

$$(12x^2z-35y)+(30x^2-14yz)$$

has a common factor of 2 on the second group, but this grouping cannot be factored as much as either of the other two groupings.

c) We first consider this grouping:

$$(27ax + 18ay + 36az) + (15x + 10y + 20z)$$

The group $(27ax + 18ay + 36az)$ has a common factor of $9a$. The group $(15x + 10y + 20z)$ has a common factor of 5. We do common factoring on each of these groups separately:

$$\begin{aligned}
&\frac{9a}{9a}(27ax + 18ay + 36az) \\
&= 9a\left(\frac{27ax}{9a} + \frac{18ay}{9a} + \frac{36az}{9a}\right) \\
&= 9a(3x + 2y + 4z)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{5}{5}(15x + 10y + 20z) \\
&= 5\left(\frac{15x}{5} + \frac{10y}{5} + \frac{20z}{5}\right) \\
&= 5(3x + 2y + 4z)
\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}
&27ax + 18ay + 36az + 15x + 10y + 20z \\
&= (27ax + 18ay + 36az) + (15x + 10y + 20z) \\
&= 9a(3x + 2y + 4z) + 5(3x + 2y + 4z)
\end{aligned}$$

Alternatively, we can apply this grouping:

$$(27ax + 15x) + (18ay + 10y) + (36az + 20z)$$

The group $(27ax + 15x)$ has a common factor of $3x$. The group $(18ay + 10y)$ has a common factor of $2y$. The group $(36az + 20z)$ has a common factor of $4z$. We do common factoring on each of these groups separately:

$$\begin{aligned}
&\frac{3x}{3x}(27ax + 15x) \\
&= 3x\left(\frac{27ax}{3x} + \frac{15x}{3x}\right) \\
&= 3x(9a + 5)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{2y}{2y}(18ay + 10y) \\
&= 2y\left(\frac{18ay}{2y} + \frac{10y}{2y}\right) \\
&= 2y(9a + 5),
\end{aligned}$$

and

$$\begin{aligned}&\frac{4z}{4z}(36az+20z)\\&=4z\left(\frac{36az}{4z}+\frac{20z}{4z}\right)\\&=4z(9a+5)\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}&27ax+18ay+36az+15x+10y+20z\\&=(27ax+15x)+(18ay+10y)+(36az+20z)\\&=3x(9a+5)+2y(9a+5)+4z(9a+5)\end{aligned}$$

There are multiple other groupings of this polynomial, but they cannot be factored as significantly as the previous two groupings.

d) We first consider this grouping:

$$(5xy-5ax)+(-3yz+3az)$$

The group $(5xy-5ax)$ has a common factor of $5x$. The group $(-3yz+3az)$ has a common factor of $3z$. We do common factoring on each of these groups separately:

$$\begin{aligned}&\frac{5x}{5x}(5xy-5ax)\\&=5x\left(\frac{5xy}{5x}+\frac{-5ax}{5x}\right)\\&=5x(y-a)\end{aligned}$$

and

$$\begin{aligned}&\frac{3z}{3z}(-3yz+3az)\\&=3z\left(\frac{-3yz}{3z}+\frac{3az}{3z}\right)\\&=3z(-y+a)\end{aligned}$$

Putting this together, we have:

$$\begin{aligned}&5xy-5ax-3yz+3az\\&=(5xy-5ax)+(-3yz+3az)\\&=5x(y-a)+3z(-y+a)\end{aligned}$$

However, we could have equally factored out $-3z$ from $(-3yz + 3az)$. In this case, we have:

$$\begin{aligned}
&\frac{-3z}{-3z}(-3yz + 3az) \\
=& -3z\left(\frac{-3yz}{-3z} + \frac{3az}{-3z}\right) \\
=& -3z(y - a)
\end{aligned}$$

This gives an overall result of:

$$\begin{aligned}
&5xy - 5ax - 3yz + 3az \\
&= (5xy - 5ax) + (-3yz + 3az) \\
&= 5x(y - a) - 3z(y - a)
\end{aligned}$$

This is a better way to factor this group, as the $(y - a)$ term matches in both groups. We will see why this is a good thing later.

Alternatively, we can apply this grouping:

$$(5xy - 3yz) + (-5ax + 3az)$$

The group $(5xy-3yz)$ has a common factor of y. The group $(-5ax+3az)$ has a common factor of a or $-a$. As with the previous grouping, it is not immediately obvious that one is better than the other. We do common factoring on each of these groups separately:

$$\begin{aligned}
&\frac{y}{y}(5xy - 3yz) \\
&= y\left(\frac{5xy}{y} + \frac{-3yz}{y}\right) \\
&= y(5x - 3z)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{a}{a}(-5ax + 3az) \\
&= a\left(\frac{-5ax}{a} + \frac{3az}{a}\right) \\
&= a(-5x + 3z)
\end{aligned}$$

or

$$\begin{aligned}
&\frac{-a}{-a}(-5ax + 3az) \\
=& -a\left(\frac{-5ax}{-a} + \frac{3az}{-a}\right) \\
=& -a(5x - 3z)
\end{aligned}$$

Putting this together, we have:

$$\begin{aligned} &5xy - 5ax - 3yz + 3az \\ &= (5xy - 3yz) + (-5ax + 3az) \\ &= y(5x - 3z) + a(-5x + 3z) \end{aligned}$$

or

$$\begin{aligned} &5xy - 5ax - 3yz + 3az \\ &= (5xy - 3yz) + (-5ax + 3az) \\ &= y(5x - 3z) - a(5x - 3z) \end{aligned}$$

This second option is a better factoring as the term $(5x - 3z)$ is shared by both groups. We will see why this is beneficial later.

This final grouping:

$$(5xy + 3az) + (-5ax - 3yz)$$

has a common factor of 2 on the second group, but this grouping cannot be factored as much as either of the other two groupings.

Solutions to problem set 2

a) We first consider this grouping:

$$\begin{aligned} &(21xy + 14x) + (3y + 2) \\ &= 7x(3y + 2) + 1(3y + 2) \end{aligned}$$

We perform common factoring, factoring the shared factor $(3y + 2)$:

$$\begin{aligned} &\frac{(3y+2)}{(3y+2)}\left[7x(3y+2) + 1(3y+2)\right] \\ &= (3y+2)\left[\frac{7x(3y+2)}{(3y+2)} + \frac{1(3y+2)}{(3y+2)}\right] \\ &= (3y+2)(7x+1) \end{aligned}$$

Next we consider the alternate grouping:

$$\begin{aligned} &(21xy + 3y) + (14x + 2) \\ &= 3y(7x + 1) + 2(7x + 1) \end{aligned}$$

We perform common factoring, factoring the shared factor $(7x+1)$:

$$\begin{aligned}&\frac{(7x+1)}{(7x+1)}\left[3y(7x+1)+2(7x+1)\right]\\&=(7x+1)\left[\frac{3y(7x+1)}{(7x+1)}+\frac{2(7x+1)}{(7x+1)}\right]\\&=(7x+1)(3y+2)\end{aligned}$$

Both groupings yield the same fully factored result.

b) We first consider this grouping:

$$\begin{aligned}&(12x^2z+30x^2)+(-8yz-20y)\\&=6x^2(2z+5)-7y(2z+5)\end{aligned}$$

We perform common factoring, factoring the shared factor $(2z+5)$:

$$\begin{aligned}&\frac{(2z+5)}{(2z+5)}\left[6x^2(2z+5)-7y(2z+5)\right]\\&=(2z+5)\left[\frac{6x^2(2z+5)}{(2z+5)}+\frac{-7y(2z+5)}{(2z+5)}\right]\\&=(2z+5)(6x^2-7y)\end{aligned}$$

Next we consider the alternate grouping:

$$\begin{aligned}&(12x^2z-14yz)+(30x^2-35y)\\&=2z(6x^2-7y)+5(6x^2-7y)\end{aligned}$$

We perform common factoring, factoring the shared factor $(6x^2-7y)$:

$$\begin{aligned}&\frac{(6x^2-7y)}{(6x^2-7y)}\left[2z(6x^2-7y)+5(6x^2-7y)\right]\\&=(6x^2-7y)\left[\frac{2z(6x^2-7y)}{(6x^2-7y)}+\frac{5(6x^2-7y)}{(6x^2-7y)}\right]\\&=(6x^2-7y)(2z+5)\end{aligned}$$

Both groupings yield the same fully factored result.

c) We first consider this grouping:

$$\begin{aligned}&(27ax+18ay+36az)+(15x+10y+20z)\\&=9a(3x+2y+4z)+5(3x+2y+4z)\end{aligned}$$

We perform common factoring, factoring the shared factor $(3x + 2y + 4z)$:

$$\begin{aligned}
&\frac{(3x+2y+4z)}{(3x+2y+4z)}\left[9a(3x+2y+4z)+5(3x+2y+4z)\right] \\
&= (3x+2y+4z)\left[\frac{9a(3x+2y+4z)}{(3x+2y+4z)}+\frac{5(3x+2y+4z)}{(3x+2y+4z)}\right] \\
&= (3x+2y+4z)(9a+5)
\end{aligned}$$

Next we consider the alternate grouping:

$$\begin{aligned}
&(27ax+15x)+(18ay+10y)+(36az+20z) \\
&= 3x(9a+5)+2y(9a+5)+4z(9a+5)
\end{aligned}$$

We perform common factoring, factoring the shared factor $(9a + 5)$:

$$\begin{aligned}
&\frac{(9a+5)}{(9a+5)}\left[3x(9a+5)+2y(9a+5)+4z(9a+5)\right] \\
&= (9a+5)\left[\frac{3x(9a+5)}{(9a+5)}+\frac{2y(9a+5)}{(9a+5)}+\frac{4z(9a+5)}{(9a+5)}\right] \\
&= (9a+5)(3x+2y+4z)
\end{aligned}$$

Both groupings yield the same fully factored result.

d) We first consider this grouping:

$$\begin{aligned}
&(5xy-5ax)+(-3yz+3az) \\
&= 5x(y-a)-3z(y-a)
\end{aligned}$$

We see now why we it was useful to identify two possible factors ($3z$ and $-3z$), and why we selected factored form with matching values in parentheses here–it allows us to perform common factoring. We perform common factoring, factoring the shared factor $(y - a)$:

$$\begin{aligned}
&\frac{(y-a)}{(y-a)}\left[5x(y-a)-3z(y-a)\right] \\
&= (y-a)\left[\frac{5x(y-a)}{(y-a)}+\frac{-3z(y-a)}{(y-a)}\right] \\
&= (y-a)(5x-3z)
\end{aligned}$$

Next we consider the alternate grouping:

$$\begin{aligned}
&(5xy-3yz)+(-5ax+3az) \\
&= y(5x-3z)-a(5x-3z)
\end{aligned}$$

Again there were two factors we could choose from (a and $-a$) yielding two factored forms. We have again selected the factored form with matching terms to allow for common factoring. We perform common factoring, factoring the shared multiple $(5x-3z)$:

$$\begin{aligned}
&\frac{(5x-3z)}{(5x-3z)}\left[y(5x-3z)-a(5x-3z)\right] \\
&=(5x-3z)\left[\frac{y(5x-3z)}{(5x-3z)}+\frac{-a(5x-3z)}{(5x-3z)}\right] \\
&=(5x-3z)(y-a)
\end{aligned}$$

Both groupings yield the same fully factored result.

Solutions to problem set 3

a) If we factor by grouping first:

$$\begin{aligned}
&12xy-15xz+36y-45z \\
&=(12xy-15xz)+(36y-45z) \\
&=\frac{3x}{3x}(12xy-15xz)+\frac{9}{9}(36y-45z) \\
&=3x\left(\frac{12xy}{3x}+\frac{-15xz}{3x}\right)+9\left(\frac{36y}{9}+\frac{-45z}{9}\right) \\
&=3x(4y-5z)+9(4y-5z) \\
&=(4y-5z)(3x+9) \\
&=(4y-5z)\frac{3}{3}(3x+9) \\
&=(4y-5z)3\left(\frac{3x}{3}+\frac{9}{3}\right) \\
&=(4y-5z)3(x+3) \\
&=3(x+3)(4y-5z)
\end{aligned}$$

Or we can group the other way:

$$\begin{aligned}
&12xy - 15xz + 36y - 45z \\
&= (12xy + 36y) + (-15xz - 45z) \\
&= \frac{12y}{12y}(12xy + 36y) + \frac{-15z}{-15z}(-15xz - 45z) \\
&= 12y\left(\frac{12xy}{12y} + \frac{36y}{12y}\right) - 15z\left(\frac{-15xz}{-15z} + \frac{-45z}{-15z}\right) \\
&= 12y(x+3) - 15z(x+3) \\
&= (x+3)(12y - 15z) \\
&= (x+3)\frac{3}{3}(12y - 15z) \\
&= (x+3)3\left(\frac{12y}{3} + \frac{-15z}{3}\right) \\
&= (x+3)3(4y - 5z) \\
&= 3(x+3)(4y - 5z)
\end{aligned}$$

Or we can perform the common factoring first:

$$\begin{aligned}
&12xy - 15xz + 36y - 45z \\
&= \frac{3}{3}(12xy - 15xz + 36y - 45z) \\
&= 3\left(\frac{12xy}{3} + \frac{-15xz}{3} + \frac{36y}{3} + \frac{-45z}{3}\right) \\
&= 3(4xy - 5xz + 12y - 15z)
\end{aligned}$$

Following this, we group the terms one way:

$$\begin{aligned}
&3(4xy - 5xz + 12y - 15z) \\
&= 3[(4xy - 5xz) + (12y - 15z)] \\
&= 3\left[\frac{x}{x}(4xy - 5xz) + \frac{3}{3}(12y - 15z)\right] \\
&= 3\left[x\left(\frac{4xy}{x} + \frac{-5xz}{x}\right) + 3\left(\frac{12y}{3} + \frac{-15z}{3}\right)\right] \\
&= 3\left[x(4y - 5z) + 3(4y - 5z)\right] \\
&= 3\left[(4y - 5z)(x+3)\right] \\
&= 3(x+3)(4y - 5z)
\end{aligned}$$

Or we can group the other way:

$$\begin{aligned}
&3(4xy - 5xz + 12y - 15z)\\
&= 3[(4xy + 12y) + (-5xz - 15z)]\\
&= 3\left[\frac{4y}{4y}(4xy + 12y) + \frac{-5z}{-5z}(-5xz - 15z)\right]\\
&= 3\left[4y\left(\frac{4xy}{4y} + \frac{12y}{4y}\right) - 5z\left(\frac{-5xz}{-5z} + \frac{-15z}{-5z}\right)\right]\\
&= 3\left[4y(x+3) - 5z(x+3)\right]\\
&= 3\left[(x+3)(4y-5z)\right]\\
&= 3(x+3)(4y-5z)
\end{aligned}$$

All four strategies give us the same result.

b) If we factoring by grouping first:

$$\begin{aligned}
&20xz + 45x + 12xyz + 27xy\\
&= (20xz + 45x) + (12xyz + 27xy)\\
&= \frac{5x}{5x}(20xz + 45x) + \frac{3xy}{3xy}(12xyz + 27xy)\\
&= 5x\left(\frac{20xz}{5x} + \frac{45x}{5x}\right) + 3xy\left(\frac{12xyz}{3xy} + \frac{27xy}{3xy}\right)\\
&= 5x(4z+9) + 3xy(4z+9)\\
&= (4z+9)(5x+3xy)\\
&= (4z+9)\frac{x}{x}(5x+3xy)\\
&= (4z+9)x\left(\frac{5x}{x} + \frac{3xy}{x}\right)\\
&= (4z+9)x(5+3y)\\
&= x(3y+5)(4z+9)
\end{aligned}$$

Or we can group the other way:

$$\begin{aligned}
&20xz + 45x + 12xyz + 27xy \\
&= (20xz + 12xyz) + (45x + 27xy) \\
&= \frac{4xz}{4xz}(20xz + 12xyz) + \frac{9x}{9x}(45x + 27xy) \\
&= 4xz\left(\frac{20xz}{4xz} + \frac{12xyz}{4xz}\right) + 9x\left(\frac{45x}{9x} + \frac{27xy}{9x}\right) \\
&= 4xz(5 + 3y) + 9x(5 + 3y) \\
&= (5 + 3y)(4xz + 9x) \\
&= (5 + 3y)\frac{x}{x}(4xz + 9x) \\
&= (5 + 3y)x\left(\frac{4xz}{x} + \frac{9x}{x}\right) \\
&= (5 + 3y)x(4z + 9) \\
&= x(3y + 5)(4z + 9)
\end{aligned}$$

Or we can perform the common factoring first:

$$\begin{aligned}
&20xz + 45x + 12xyz + 27xy \\
&= \frac{x}{x}(20xz + 45x + 12xyz + 27xy) \\
&= x\left(\frac{20xz}{x} + \frac{45x}{x} + \frac{12xyz}{x} + \frac{27xy}{x}\right) \\
&= x(20z + 45 + 12yz + 27y)
\end{aligned}$$

Following this, we group the terms one way:

$$\begin{aligned}
&x(20z + 45 + 12yz + 27y) \\
&= x[(20z + 45) + (12yz + 27y)] \\
&= x\left[\frac{5}{5}(20z + 45) + \frac{3y}{3y}(12yz + 27y)\right] \\
&= x\left[5\left(\frac{20z}{5} + \frac{45}{5}\right) + 3y\left(\frac{12yz}{3y} + \frac{27y}{3y}\right)\right] \\
&= x\left[5(4z + 9) + 3y(4z + 9)\right] \\
&= x\left[(4z + 9)(5 + 3y)\right] \\
&= x(3y + 5)(4z + 9)
\end{aligned}$$

Or we can group the other way:

$$
\begin{aligned}
& x(20z + 45 + 12yz + 27y) \\
&= x[(20z + 12yz) + (45 + 27y)] \\
&= x\left[\frac{4z}{4z}(20z + 12yz) + \frac{9}{9}(45 + 27y)\right] \\
&= x\left[4z\left(\frac{20z}{4z} + \frac{12yz}{4z}\right) + 9\left(\frac{45}{9} + \frac{27y}{9}\right)\right] \\
&= x\left[4z(5 + 3y) + 9(5 + 3y)\right] \\
&= x\left[(5 + 3y)(4z + 9)\right] \\
&= x(3y + 5)(4z + 9)
\end{aligned}
$$

All four strategies give us the same result.